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DIFFRACTION OF A DIPOLE FIELD BY A CONICAL FIELD+

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Summary

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The diffracted field due to a perfectly conducting conical ring, in the presence of an axially oriented electric dipole located on the axis of symmetry $\theta = 0$, is obtained by means of an analytic continuation technique.

The conical ring occupies that region $\alpha < \theta < \pi - \alpha$ of a spherical shell $b < r < a$, which is limited by the surface of a cone of semi-angle $\theta = \alpha$.

Various particular cases are investigated in detail, and the solutions of other diffraction problems by the same continuation technique are outlined.

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1. Introduction

In this paper, the diffracted electromagnetic field produced by an axially oriented electric dipole located on the axis of symmetry of a perfectly conducting conical ring is considered. The free space surrounding the ring is divided into various regions, and in each region the components of the diffracted electromagnetic field are derived from a properly chosen Hertzian function. The unknown coefficients which appear in the expressions of these Hertzian functions are then determined by requiring that the components of the total electromagnetic field satisfy the boundary conditions on the surface of the ring, and be continuous across the ideal surfaces which separate two adjacent regions of space.

Although it appears that this boundary value problem has not been previously considered, the method of solution, which makes use of Legendre functions of non-integral order and of an appropriate continuation technique, was firstly employed by Schelkunoff (1941) in his treatment of the biconical antenna, and subsequently by many other authors (see e.g. Northover (1962) and Rogers, Schindler and Schultz (1963)).

In the following, the rationalized MKS system of units is used, and the time-dependence factor $e^{-i\omega t}$ is omitted.

2. Statement of the problem and boundary conditions.

With reference to a system of spherical polar coordinates (r, θ, ϕ), the perfectly conducting conical ring occupies that region $\alpha < \theta < \pi - \alpha$ of a spherical shell $b < r < a$, which is limited by the surface of a cone of semi-angle $\theta = \alpha$

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(fig. 1 shows a cross section of the scatterer in a plane through the axis of symmetry). An axially oriented electric dipole is located on the axis of symmetry of the ring at a point $\theta = 0$, $r = h$.

The components of the diffracted (incident plus scattered) electric field $\vec{E}$ and of the diffracted magnetic field $\vec{H}$ in the free space surrounding the ring can be derived from a scalar function $U(r, \theta)$ by means of the relations (Watson, 1918; Fock, 1945):

$$E_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial r} \left(\sin\theta \frac{\partial U}{\partial \theta}\right), \tag{1-1}$$

$$E_\theta = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial \theta}\right), \tag{1-2}$$

$$H_\phi = -\frac{ik}{Z} \frac{\partial U}{\partial \theta}, \tag{1-3}$$

$$E_\phi = H_r = H_\theta = 0, \tag{1-4}$$

where $k = \omega \sqrt{\epsilon_o \mu_o}$ and $Z = \sqrt{\mu_o / \epsilon_o} = 120\ \pi$ ohms are respectively the free space wave number and the free space intrinsic impedance. The Hertzian function U is independent of the ϕ coordinate, due to the axial symmetry of the problem, and must satisfy the reduced wave equation

$$(\nabla^2 + k^2) U = 0 \tag{2}$$

the radiation condition at infinity, and the appropriate boundary conditions at the ring's surface.

Since the conical ring is perfectly conductive, the tangential components of the total electric field at its surface are zero. On the slant surfaces $\theta = \alpha$, $\theta = \pi - \alpha$ of the ring $E_r = 0$ or, using relations (1-1) and (2):

$$\left(\frac{\partial^2}{\partial r^2} + k^2\right)(r\,U) = 0\,, \tag{3}$$

which implies that on the slant surfaces

$$U = r^{-1}(A \sin kr + B \cos kr), \tag{4}$$

where A and B are constants.

On the bases $r = a$ and $r = b$ of the ring $E_\theta = 0$, or:

$$\frac{\partial}{\partial\theta}\left\{\frac{\partial(r\,U)}{\partial r}\right\} = 0\,, \tag{5}$$

which implies that

$$\begin{aligned} \frac{\partial(rU)}{\partial r} &= M_1\,, \quad \text{on the basis } r = a\,, \\ &= M_2\,, \quad \text{on the basis } r = b\,, \end{aligned} \tag{6}$$

where M_1 and M_2 are constants.

Let us consider the function

$$\tilde{U}(r,\theta) = U(r,\theta) - r^{-1}(A \sin kr + B \cos kr)\,, \tag{7}$$

which does not change the field components if used instead of U in relations (1), and which satisfies the reduced wave equation if U satisfies (2). On the slant surfaces $\tilde{U} = 0$; on the bases $\frac{\partial(r\tilde{U})}{\partial r} = 0$, provided that the arbitrary constants are chosen to satisfy the relations:

$$M_1 = k(A \cos ka - B \sin ka)\,, \tag{8-1}$$

$$M_2 = k(A \cos kb - B \sin kb) \tag{8-2}$$

In conclusion, suppressing the superscript $\sim$, we may impose the boundary condition $U = 0$ on the slant surfaces $\theta = \alpha$ and $\theta = \pi - \alpha$, and the boundary condition $\frac{\partial(rU)}{\partial r} = 0$ on the spherical bases $r = a$ and $r = b$ of the conical ring.

The primary field due to the electric dipole may be derived from the function

$$U_o = V \frac{e^{ikR}}{kR} , \tag{9}$$

where V is a constant with the dimensions of an electric voltage, and

$$R = (r^2 + h^2 - 2hr\eta)^{1/2} , \quad \eta = \cos\theta . \tag{10}$$

The function U_o can be expanded in the form

$$\begin{aligned} U_o &= \frac{iV}{kh}\frac{1}{kr}\sum_{n=0}^{\infty}(2n+1)\zeta_n(kh)\psi_n(kr)P_n(\eta), \qquad \text{for } r < h , \\ &= \frac{iV}{kh}\frac{1}{kr}\sum_{n=0}^{\infty}(2n+1)\psi_n(kh)\zeta_n(kr)P_n(\eta), \qquad \text{for } r > h , \end{aligned} \tag{11}$$

where $P_n(\eta)$ is the Legendre polynominal of degree n and order zero, and ψ_n and ζ_n are related to the Bessel and Hankel functions by the expressions:

$$\psi_n(x) = \sqrt{\frac{\pi x}{2}}\, J_{n+\frac{1}{2}}(x) , \qquad \zeta_n(x) = \sqrt{\frac{\pi x}{2}}\, H^{(1)}_{n+\frac{1}{2}}(x) . \tag{12}$$

In the following, the exact diffracted electromagnetic field is obtained by dividing the space around the ring in various regions and by determining the appropriate Hertzian function for each region. The analysis is carried out in detail for the two cases $0 < h < b$ (in sections 3 and 4) and $h > a$ (in sections 5 and 6), while further extensions of the method are outlined in section 7. In sections 3 and 4, the case in which the dipole lies at the origin $r = 0$ is explicitly excluded in order to avoid difficulties (Kleinman and Senior, 1963).

3. General solution for the case $0 < h < b$.

The free space surrounding the conical ring is divided into the following four regions:

Region I $(r \leqslant b)$,

Region II $(b \leqslant r \leqslant a;\ 0 \leqslant \theta < \alpha)$,

Region III $(b \leqslant r \leqslant a;\ \pi - \alpha < \theta \leqslant \pi)$,

Region IV $(r \geqslant a)$.

The Hertzian functions for these four regions may be expanded into infinite series of elementary wave functions. Since the scattered field must be finite at all points of the axis of symmetry and must satisfy the radiation condition at infinity, it is found that in region I:

$$U = U_1 = U_o + \frac{iV}{kh}\,\frac{1}{kr}\sum_{n=1}^{\infty} A_n \psi_n(kr)\, P_n(\eta) \; ; \tag{13}$$

in region II:

$$U = U_2 = \frac{iV}{kh}\,\frac{1}{kr}\sum_{j=1}^{\infty}\left\{C_j \psi_{\beta_j}(kr) + D_j \zeta_{\beta_j}(kr)\right\} P_{\beta_j}(\eta) \; ; \tag{14}$$

in region III:

$$U = U_3 = \frac{iV}{kh}\,\frac{1}{kr}\sum_{j=1}^{\infty}\left\{\widetilde{C}_j \psi_{\beta_j}(kr) + \widetilde{D}_j \zeta_{\beta_j}(kr)\right\} P_{\beta_j}(-\eta) \; ; \tag{15}$$

and in region IV:

$$U = U_4 = \frac{iV}{kh}\,\frac{1}{kr}\sum_{n=1}^{\infty} B_n \zeta_n(kr)\, P_n(\eta) \; ; \tag{16}$$

where the positive numbers β_j are given by the equation:

$$P_{\beta_j}(\eta_o) = 0 \,, \quad \eta_o = \cos\alpha \,. \tag{17}$$

The series expansions (13) and (16) should contain terms corresponding to n = 0; however, these terms give no contribution to the diffracted field components, as it easily follows from formulas (1) and from the relation $P_o(\eta) = 1$. Therefore the coefficients A_o and B_o may be chosen arbitrarily, and we set $A_o = B_o = 0$; similarly, the first term of expansion (11) may be neglected without any loss of generality. In the following, n will be a positive integer.

The coefficients A_n, B_n, C_j, D_j, $\widetilde{C}_j$ and $\widetilde{D}_j$ are determined by imposing the boundary conditions on the surface of the conical ring and the continuity of the diffracted field components across the ideal surfaces which separate the four regions from one another. The boundary conditions on the slant surfaces $\theta = \alpha$, $\theta = \pi - \alpha$ are automatically satisfied by the expansions (14) and (15). In order that $E_\theta = 0$ on the spherical bases of the scatterer, it must be:

$$\left\{\frac{\partial(rU_1)}{\partial r}\right\}_{r=b} = 0 \,, \qquad \text{for } \alpha < \theta < \pi - \alpha \,, \tag{18}$$

$$\left\{\frac{\partial(rU_4)}{\partial r}\right\}_{r=a} = 0 \qquad \text{for } \alpha < \theta < \pi - \alpha \,, \tag{19}$$

while the continuity of E_r, E_θ and H_ϕ across the surfaces $r = a$ and $r = b$ in the angular ranges $0 \leqslant \theta < \alpha$ and $\pi - \alpha < \theta \leqslant \pi$ is guaranteed by the conditions:

$$\left\{\frac{\partial(rU_1)}{\partial r}\right\}_{r=b} = \left\{\frac{\partial(rU_2)}{\partial r}\right\}_{r=b}, \qquad \text{for } 0 \leqslant \theta < \alpha,$$

$$= \left\{\frac{\partial(rU_3)}{\partial r}\right\}_{r=b}, \qquad \text{for } \pi - \alpha < \theta \leqslant \pi; \tag{20}$$

$$\left\{\frac{\partial(rU_4)}{\partial r}\right\}_{r=a} = \left\{\frac{\partial(rU_2)}{\partial r}\right\}_{r=a}, \qquad \text{for } 0 \leqslant \theta < \alpha,$$

$$\left\{\frac{\partial(rU_3)}{\partial r}\right\}_{r=a}, \qquad \text{for } \pi - \alpha < \theta \leqslant \pi; \tag{21}$$

$$U_1(b, \theta) = U_2(b, \theta), \qquad \text{for } 0 \leqslant \theta < \alpha,$$

$$= U_3(b, \theta), \qquad \text{for } \pi - \alpha < \theta \leqslant \pi; \tag{22}$$

$$U_4(a, \theta) = U_2(a, \theta), \qquad \text{for } 0 \leqslant \theta < \alpha,$$

$$= U_3(a, \theta), \qquad \text{for } \pi - \alpha < \theta \leqslant \pi. \tag{23}$$

The unknown coefficients are obtained from relations (18) to (23) by making use of the orthogonality properties of the Legendre functions and of the Wronskian relation for the Bessel functions. Since this technique has been illustrated in detail by other authors (see e. g. Northover, 1962), only the results are given here. Let us set:

$$C_j^{\pm} = \frac{1}{2}(C_j \pm \tilde{C}_j), \qquad D_j^{\pm} = \frac{1}{2}(D_j \pm \tilde{D}_j); \tag{24}$$

then one finds:

$$A_n = -\frac{2n+1}{\psi_n'(kb)}\left[\psi_n(kh)\,\zeta_n'(kb) + \sum_{j=1}^{\infty} f_n(\beta_j)\left\{C_j^{\pm}\,\psi_{\beta_j}'(kb) + D_j^{\pm}\,\zeta_{\beta_j}'(kb)\right\}\right], \tag{25}$$

$$B_n = -\frac{2n+1}{\zeta_n'(ka)} \sum_{j=1}^{\infty} f_n(\beta_j) \left\{ C_j^{\pm} \psi_{\beta_j}'(ka) + D_j^{\pm} \zeta_{\beta_j}'(ka) \right\}, \tag{26}$$

where C_j^+ and D_j^+ are to be used for n even, C_j^- and D_j^- for n odd. and the primes indicate derivatives with respect to the argument ka or kb. The coefficients $C_j^{\pm}$ and $D_j^{\pm}$ are given by the equations:

$$C_j^+ \psi_{\beta_j}(ka) + D_j^+ \zeta_{\beta_j}(ka) = -\sum_{m=1}^{\infty} (4m+1) \frac{f_{2m}(\beta_j)}{f(\beta_j)} \frac{\zeta_{2m}(ka)}{\zeta_{2m}'(ka)} \times$$

$$\times \sum_{\ell=1}^{\infty} f_{2m}(\beta_\ell) \left\{ C_\ell^+ \psi_{\beta_\ell}'(ka) + D_\ell^+ \zeta_{\beta_\ell}'(ka) \right\}, \tag{27-1}$$

$$C_j^+ \psi_{\beta_j}(kb) + D_j^+ \zeta_{\beta_j}(kb) = -\sum_{m=1}^{\infty} \frac{4m+1}{\psi_{2m}'(kb)} \frac{f_{2m}(\beta_j)}{f(\beta_j)} \times$$

$$\times \left[i\psi_{2m}(kh) + \psi_{2m}(kb) \sum_{\ell=1}^{\infty} f_{2m}(\beta_\ell) \left\{ C_\ell^+ \psi_{\beta_\ell}'(kb) + D_\ell^+ \zeta_{\beta_\ell}'(kb) \right\} \right], \tag{27-2}$$

$$C_j^- \psi_{\beta_j}(ka) + D_j^- \zeta_{\beta_j}(ka) = -\sum_{m=1}^{\infty} (4m-1) \frac{f_{2m-1}(\beta_j)}{f(\beta_j)} \frac{\zeta_{2m-1}(ka)}{\zeta_{2m-1}'(ka)} \times$$

$$\times \sum_{\ell=1}^{\infty} f_{2m-1}(\beta_\ell) \left\{ C_\ell^- \psi_{\beta_\ell}'(ka) + D_\ell^- \zeta_{\beta_\ell}'(ka) \right\}, \tag{28-1}$$

$$C_j^- \psi_{\beta_j}(kb) + D_j^- \zeta_{\beta_j}(kb) = -\sum_{m=1}^{\infty} \frac{4m-1}{\psi_{2m-1}'(kb)} \frac{f_{2m-1}(\beta_j)}{f(\beta_j)} \times$$

$$\times \left[i\psi_{2m-1}(kh) - \psi_{2m-1}(kb) \sum_{\ell=1}^{\infty} f_{2m-1}(\beta_\ell) \left\{ C_\ell^- \psi_{\beta_\ell}'(kb) + D_\ell^- \zeta_{\beta_\ell}'(kb) \right\} \right], \tag{28-2}$$

where:

$$f(\beta_j) = \int_1^{\eta_o} \left\{ P_{\beta_j}(\eta) \right\}^2 d\eta \; . \tag{29}$$

$$f_n(\beta_j) = \int_1^{\eta_o} P_{\beta_j}(\eta)\, P_n(\eta)\, d\eta . \tag{30}$$

The coefficients C_j^+ and D_j^+ are computed by means of relations (27), C_j^- and D_j^- by means of (28), then C_j, $\widetilde{C}_j$, D_j and $\widetilde{D}_j$ are obtained from (24), and A_n and B_n from (25) and (26). Thus, the problem of the exact determination of the diffracted field is reduced to the problem of solving the two systems (27) and (28) of infinite linear algebraic equations in an infinite number of unknowns.

4. Particular solutions for the case $0 < h < b$.

In this section, four particular solutions corresponding to special values of the parameters a, b and α are considered, for the case $0 < h < b$.

Case $a = b$.

The geometry of the scatterer is shown in fig. 2. It is found that

$$A_n = 0, \qquad B_n = (2n+1)\,\psi_n(kh), \tag{31}$$

and therefore one has the primary field only. This result should have been expected, since when $a = b$ regions II and III vanish. In the limit $a = b$, there is no conical ring: one simply has a dipole in free space.

Case $\alpha = \pi/2$.

The scatterer is now a perfectly conducting annular disc of zero thickness (fig. 3). It is found that:

$$\beta_j = 2j - 1, \qquad j = 1, 2, \ldots, \tag{32}$$

and therefore:

$$f(2j-1) = -\frac{1}{4j-1}, \tag{33}$$

$$f_{2m}(2j-1) = \frac{(-1)^{m+j+1}(2m)!\,(2j-1)!}{2^{2m+2j-1}(2m-2j+1)(m+j)\left[m!\,(j-1)!\right]^2}, \tag{34-1}$$

$$f_{2m-1}(2j-1) = -\frac{1}{4j-1}\,\delta_{jm}, \tag{34-2}$$

where δ_{jm} is the Kronecker symbol. It follows that:

$$A_{2m-1} = 0, \tag{35-1}$$

$$A_{2m} = -\frac{4m+1}{\psi'_{2m}(kb)}\left[\psi_{2m}(kh)\zeta'_{2m}(kb) + \right.$$

$$\left. + \sum_{j=1}^{\infty} f_{2m}(2j-1)\left\{C_j\,\psi'_{2j-1}(kb) + D_j^{+}\,\zeta'_{2j-1}(kb)\right\}\right], \tag{35-2}$$

$$B_{2m-1} = (4m-1)\psi_{2m-1}(kh), \tag{36-1}$$

$$B_{2m} = -\frac{4m+1}{\zeta'_{2m}(ka)}\sum_{j=1}^{\infty} f_{2m}(2j-1)\left\{C_j\psi'_{2j-1}(ka) + D_j^{+}\,\zeta'_{2j-1}(ka)\right\}, \tag{36-2}$$

$$C_j^{-} = 0, \tag{37-1}$$

$$D_j^{-} = (4j-1)\psi_{2j-1}(kh), \tag{37-2}$$

While the coefficients C_j and D_j^{+} are given by the system:

$$C_j \psi_{2j-1}(kb) + D_j^+ \zeta_{2j-1}(kb) = -\sum_{m=1}^{\infty} \frac{4m+1}{\psi'_{2m}(kb)} \frac{f_{2m}(2j-1)}{f(2j-1)} \Big[i\psi_{2m}(kh) +$$

$$+ \psi_{2m}(kb) \sum_{\ell=1}^{\infty} f_{2m}(2\ell-1) \left\{ C_\ell \psi'_{2\ell-1}(kb) + D_\ell^+ \zeta'_{2\ell-1}(kb) \right\} \Big] ,$$

(38-1)

$$C_j \psi_{2j-1}(ka) + D_j^+ \zeta_{2j-1}(ka) = -\sum_{m=1}^{\infty} (4m+1) \frac{f_{2m}(2j-1)}{f(2j-1)} \frac{\zeta_{2m}(ka)}{\zeta'_{2m}(ka)} \times$$

(38-2)

$$\times \sum_{\ell=1}^{\infty} f_{2m}(2\ell-1) \left\{ C_\ell \psi'_{2\ell-1}(ka) + D_\ell^+ \zeta'_{2\ell-1}(ka) \right\}$$

Case $a = \infty$.

Region IV disappears (fig. 4), and in order to satisfy the radiation condition,

$$C_j = 0, \qquad \tilde{C}_j = 0 . \tag{39}$$

The coefficients A_n, D_j and $\tilde{D}_j$ are given by the relations:

$$D_j^+ = -\frac{1}{\zeta_{\beta_j}(kb)} \sum_{m=1}^{\infty} \frac{4m+1}{\psi'_{2m}(kb)} \frac{f_{2m}(\beta_j)}{f(\beta_j)} \Big[i\,\psi_{2m}(kh) +$$

(40)

$$+ \psi_{2m}(kb) \sum_{\ell=1}^{\infty} f_{2m}(\beta_\ell)\, \zeta'_{\beta_\ell}(kb)\, D_\ell^+ \Big] ,$$

$$D_j^- = -\frac{1}{\zeta_{\beta_j}(kb)} \sum_{m=1}^{\infty} \frac{4m-1}{\psi'_{2m-1}(kb)} \frac{f_{2m-1}(\beta_j)}{f(\beta_j)} \Big[i\psi_{2m-1}(kh) + $$

$$+ \psi_{2m-1}(kb) \sum_{\ell=1}^{\infty} f_{2m-1}(\beta_\ell)\, \zeta'_{\beta_\ell}(kb)\, D_\ell^- \Big] , \tag{41}$$

$$A_n = -\frac{2n+1}{\psi'_n(kb)} \Big[\psi_n(kh)\, \zeta'_n(kb) + \sum_{j=1}^{\infty} f_n(\beta_j)\, \zeta'_{\beta_j}(kb)\, D_j^{\pm} \Big] , \tag{42}$$

where D_j^+ is to be used for n even, and D_j^- for n odd.

<u>Case $a = \infty$, $\alpha = \pi/2$</u> .

This is the case of a circular aperture in a perfectly conducting screen (fig. 5). Formulas (39), (40), (41) and (42) hold with β_j, $f(\beta_j)$ and $f_n(\beta_j)$ given by (32), (33) and (34). In particular,

$$D_j^- = (4j-1)\, \psi_{2j-1}(kh) , \tag{43}$$

and therefore

$$A_{2m-1} = 0 \tag{44}$$

5. <u>General solution for the case $h > a$</u> .

The geometry of the problem is that shown in fig. 1, but the dipole is now located at a distance $h > a$ from the origin. The free space surrounding the conical ring is still divided into the four regions of section 3, and the Hertzian functions are now given by the expressions:

$$U_1 = \frac{iV}{kh} \frac{1}{kr} \sum_{n=1}^{\infty} a_n \psi_n(kr)\, P_n(\eta) , \tag{45}$$

$$U_2 = \frac{iV}{kh}\,\frac{1}{kr}\sum_{j=1}^{\infty}\left\{c_j\,\psi_{\beta_j}(kr) + d_j\,\zeta_{\beta_j}(kr)\right\}\,P_{\beta_j}(\eta) \tag{46}$$

$$U_3 = \frac{iV}{kh}\,\frac{1}{kr}\sum_{j=1}^{\infty}\left\{\tilde{c}_j\,\psi_{\beta_j}(kr) + d_j\,\zeta_{\beta_j}(kr)\right\}\,P_{\beta_j}(-\eta)\,, \tag{47}$$

$$U_4 = U_o + \frac{iV}{kh}\,\frac{1}{kr}\sum_{n=1}^{\infty} b_n\,\zeta_n(kr)\,P_n(\eta)\,, \tag{48}$$

in regions I, II, III and IV, respectively. The positive numbers β_j are given by equation (17), and the coefficients a_n, b_n, c_j, $\tilde{c}_j$, d_j and $\tilde{d}_j$ are determined by imposing the boundary conditions on the surface of the conical ring and the continuity of the diffracted field components across the ideal surfaces $r = a$ and $r = b$ which separate the various regions of free space from one another. It is found that:

$$a_n = -\frac{2n+1}{\psi_n'(kb)}\sum_{j=1}^{\infty} f_n(\eta_j)\left\{c_j^{\pm}\,\psi'_{\beta_j}(kb) + d_j^{\pm}\,\zeta'_{\beta_j}(kb)\right\}\,, \tag{49}$$

$$b_n = -\frac{2n+1}{\zeta_n'(ka)}\left[\zeta_n(kh)\psi_n'(ka) + \sum_{j=1}^{\infty} f_n(\beta_j)\left\{c_j^{\pm}\,\psi'_{\beta_j}(ka) + d_j^{\pm}\,\zeta'_{\beta_j}(ka)\right\}\right]\,.$$

where:

$$c_j^{\pm} = \frac{1}{2}\,(c_j \pm \tilde{c}_j)\,, \qquad d_j^{\pm} = \frac{1}{2}\,(d_j \pm \tilde{d}_j)\,, \tag{51}$$

and c_j^+ and d_j^+ are to be used for n even, c_j^- and d_j^- for n odd. The coefficients $c_j^{\pm}$ and $d_j^{\pm}$ are given by the two following systems of infinite linear algebraic equations in infinite unknowns:

$$c_j^+ \psi_{\beta_j}(kb) + d_j^+ \zeta_{\beta_j}(kb) = -\sum_{m=1}^{\infty} (4m+1) \frac{f_{2m}(\beta_j)}{f(\beta_j)} \frac{\psi_{2m}(kb)}{\psi'_{2m}(kb)} \times$$

$$\times \sum_{\ell=1}^{\infty} f_{2m}(\beta_\ell) \left\{ c_\ell^+ \psi'_{\beta_\ell}(kb) + d_\ell^+ \zeta'_{\beta_\ell}(kb) \right\}, \tag{52-1}$$

$$c_j^+ \psi_{\beta_j}(ka) + d_j^+ \zeta_{\beta_j}(ka) = \sum_{m=1}^{\infty} \frac{4m+1}{\zeta'_{2m}(ka)} \frac{f_{2m}(\beta_j)}{f(\beta_j)} \Big[i\, \zeta_{2m}(kh) -$$

$$- \zeta_{2m}(ka) \sum_{\ell=1}^{\infty} f_{2m}(\beta_\ell) \left\{ c_\ell^+ \psi'_{\beta_\ell}(ka) + d_\ell^+ \zeta'_{\beta_\ell}(ka) \right\} \Big], \tag{52-2}$$

$$c_j^- \psi_{\beta_j}(kb) + d_j^- \zeta_{\beta_j}(kb) = -\sum_{m=1}^{\infty} (4m-1) \frac{f_{2m-1}(\beta_j)}{f(\beta_j)} \frac{\psi_{2m-1}(kb)}{\psi'_{2m-1}(kb)} \times$$

$$\times \sum_{\ell=1}^{\infty} f_{2m-1}(\beta_\ell) \left\{ c_\ell^- \psi'_{\beta_\ell}(kb) + d_\ell^- \zeta'_{\beta_\ell}(kb) \right\}, \tag{53-1}$$

$$c_j^- \psi_{\beta_j}(ka) + d_j^- \zeta_{\beta_j}(ka) = \sum_{m=1}^{\infty} \frac{4m-1}{\zeta'_{2m-1}(ka)} \frac{f_{2m-1}(\beta_j)}{f(\beta_j)} \Big[i\, \zeta_{2m-1}(kh) -$$

$$- \zeta_{2m-1}(ka) \sum_{\ell=1}^{\infty} f_{2m-1}(\beta_\ell) \left\{ c_\ell^- \psi'_{\beta_\ell}(ka) + d_\ell^- \zeta'_{\beta_\ell}(ka) \right\} \Big], \tag{53-2}$$

where $f(\beta_j)$ and $f_n(\beta_j)$ are still given by (29) and (30).

6. Particular solutions for the case $h > a$.

In this section, four particular solutions corresponding to special values of the parameters a, b, and α are considered, for the case $h > a$.

Case $a = b$.

The geometry for this case is still that shown in fig. 2 but with $h > a$. One finds that

$$a_n = (2n+1)\, \zeta_n(kh)\,; \qquad b_n = 0\,, \tag{54}$$

i.e. the limit $a = b$ corresponds to a dipole in free space.

Case $\alpha = \pi/2$

The geometry for this case is illustrated in fig. 3; the dipole must now be located at a distance $h > a$ from the origin of the coordinate system. The coefficients are given by the following relations:

$$a_{2m-1} = (4m-1)\,\zeta_{2m-1}(kh)\,, \tag{55-1}$$

$$a_{2m} = -\frac{4m+1}{\psi'_{2m}(kb)} \sum_{j=1}^{\infty} f_{2m}(2j-1) \left\{ c_j^+ \psi'_{2j-1}(kb) + d_j \zeta'_{2j-1}(kb) \right\}\,, \tag{55-2}$$

$$b_{2m-1} = 0\,, \tag{56-1}$$

$$b_{2m} = -\frac{4m+1}{\zeta'_{2m}(ka)} \Big[\zeta_{2m}(kh)\,\psi'_{2m}(ka) + $$

$$+ \sum_{j=1}^{\infty} f_{2m}(2j-1) \left\{ c_j^+ \psi'_{2j-1}(ka) + d_j \zeta'_{2j-1}(ka) \right\} \Big] \tag{56-2}$$

$$c_j^- = (4j-1)\,\zeta_{2j-1}(kh)\,, \tag{57-1}$$

$$d_j^- = 0\,. \tag{57-2}$$

where the coefficients c_j^+ and d_j are given by the system:

$$c_j^+ \psi_{2j-1}(kb) + d_j \zeta_{2j-1}(kb) = -\sum_{m=1}^{\infty} (4m+1) \frac{f_{2m}(2j-1)}{f(2j-1)} \frac{\psi_{2m}(kb)}{\psi'_{2m}(kb)} \times$$

$$\times \sum_{\ell=1}^{\infty} f_{2m}(2\ell-1) \left\{ c_\ell^+ \psi'_{2\ell-1}(kb) + d_\ell \zeta'_{2\ell-1}(kb) \right\}\,, \tag{58-1}$$

$$c_j^+ \psi_{2j-1}(ka) + d_j \zeta_{2j-1}(ka) = \sum_{m=1}^{\infty} \frac{4m+1}{\zeta'_{2m}(ka)} \frac{f_{2m}(2j-1)}{f(2j-1)} \left[i\zeta_{2m}(kh) - \right.$$

$$\left. - \zeta_{2m}(ka) \sum_{\ell=1}^{\infty} f_{2m}(2\ell-1) \left\{ c_\ell^+ \psi'_{2\ell-1}(ka) + d_\ell \zeta'_{2\ell-1}(ka) \right\} \right] . \tag{58-2}$$

Case b = 0 .

Region I disappears (fig. 6), and

$$d_j = 0 , \qquad \tilde{d}_j = 0 . \tag{59}$$

The coefficients b_n, c_j and $\tilde{c}_j$ are given by:

$$c_j^+ = \frac{1}{\psi_{\beta_j}(ka)} \sum_{m=1}^{\infty} \frac{4m+1}{\zeta'_{2m}(ka)} \frac{f_{2m}(\beta_j)}{f(\beta_j)} \left[i\zeta_{2m}(kh) - \right. \tag{60}$$

$$\left. - \zeta_{2m}(ka) \sum_{\ell=1}^{\infty} f_{2m}(\beta_\ell) \psi'_{\beta_\ell}(ka) c_\ell^+ \right] ,$$

$$c_j^- = \frac{1}{\psi_{\beta_j}(ka)} \sum_{m=1}^{\infty} \frac{4m-1}{\zeta'_{2m-1}(ka)} \frac{f_{2m-1}(\beta_j)}{f(\beta_j)} \left[i\zeta_{2m-1}(kh) - \right.$$

$$\left. - \zeta_{2m-1}(ka) \sum_{\ell=1}^{\infty} f_{2m-1}(\beta_\ell) \psi'_{\beta_\ell}(ka) c_\ell^- \right] , \tag{61}$$

$$b_n = - \frac{2n+1}{\zeta'_n(ka)} \left[\zeta_n(kh) \psi'_n(ka) + \sum_{j=1}^{\infty} f_n(\beta_j) \psi'_{\beta_j}(ka) c_j^{\pm} \right] , \tag{62}$$

where c_j^+ and c_j^- are to be used for n even and odd, respectively.

Case b = 0, $\alpha = \pi/2$.

The scatterer is a perfectly conducting circular disc of zero thickness (fig. 7). The coefficients of the Hertzian functions expansions are given by formulas (59), (60), (61) and (62), in which β_j, $f(\beta_j)$ and $f_n(\beta_j)$ have the values (32), (33) and (34). In particular

$$c_j^- = (4j-1)\,\zeta_{2j-1}(kh)\,, \tag{63}$$

and therefore,

$$b_{2m-1} = 0 \quad . \tag{64}$$

7. Further applications of the method.

The technique employed in the preceding sections can also be applied when $b < h < a$, but U_o must then be expanded in terms of the functions $P_{\beta_j}(\eta)$, which are orthogonal in $0 \leqslant \theta \leqslant \alpha$.

The case in which the source is an axially oriented magnetic dipole located on the axis of symmetry of the conical ring is not very different from the electric dipole case, and is dealt with in a similar way. The diffracted field components can now be derived from a scalar function $W(r, \theta)$ through formulas:

$$H_r = \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,\frac{\partial W}{\partial\theta})\,, \tag{65-1}$$

$$H_\theta = -\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial W}{\partial\theta})\,, \tag{65-2}$$

$$E_\phi = i k Z\,\frac{\partial W}{\partial\theta}\,, \tag{65-3}$$

$$H_\phi = E_r = E_\theta = 0\,, \tag{65-4}$$

where

$$(\nabla^2 + k^2)\,W = 0\,, \tag{66}$$

and the boundary conditions reduce to $\frac{\partial W}{\partial \theta} = 0$ on the entire surface of the conical ring.

In general, the method of solution used in this paper is applicable to all diffraction problems for which:

1) The scatterer is a perfectly conducting body of revolution, whose surface is made of portions of concentric spherical surfaces and of portions of conical surfaces having their common vertex at the center of the concentric spheres;

2) The source is an electric or magnetic dipole, axially oriented and located on the axis of symmetry of the scatterer.

8. Conclusion

The diffracted electromagnetic field for the particular configuration antenna-scatterer considered in this paper has been exactly determined. However, the solution is of little practical utility as it stands, because the mode coefficients must satisfy infinite sets of linear algebraic equations that we are unable to solve.

In all practical applications, only a finite number of modes is taken into account, and numerical results are obtained with the aid of a computer. The simplest way of choosing these preferred modes is to consider only the first few terms (lower modes) of the infinite series which represent the Hertzian functions; this corresponds to replacing each infinite set of equations with a truncated set (an example of this procedure is developed in the Appendix). A better choice of the preferred set of modes might be based on the physical consideration that modes in two adjacent regions of space which are of like order will show maximum coupling to each other; this selective mode coupling was proposed and successfully employed by Plonus (1961, 1963) in his calculations of the radiation pattern of biconical antennas.

9. Acknowledgment.

The author wishes to thank Dr. T. B. A. Senior for helpful discussions.

10. Appendix.

Let us consider the case in which $h > a$ and $\alpha = \pi/2$, and let us suppose that ka is smaller than unity. Taking into account only the first two modes in regions II and III, the infinite set of linear equations (58) reduces to a truncated set of four equations in the four unknowns c_1^+, c_2^+ d_1 and d_2. If the Bessel and Hankel functions are expanded out in power series of ka, it is found that:

$$c_1^+ = \frac{5}{16}\,\zeta_2(kh)\,\frac{Q_1^+}{Q}\,ka\left\{1+0\left[(ka)^2\right]\right\} \quad , \tag{A-1}$$

$$c_2^+ = \frac{175}{16}\,\zeta_2(kh)\,\frac{Q_2^+}{Q}\,(ka)^{-1}\left\{1+0\left[(ka)^2\right]\right\} \quad , \tag{A-2}$$

$$d_1 = \frac{5i}{48}\,\zeta_2(kh)\,\frac{Q_1}{Q}\,\Delta(ka)^4\left\{1+0\left[(ka)^2\right]\right\} \quad , \tag{A-3}$$

$$d_2 = \frac{i}{144}\,\zeta_2(kh)\,\frac{Q_2}{Q}\,\Delta^3(ka)^5\left\{1+0\left[(ka)^2\right]\right\} \quad , \tag{A-4}$$

where

$$0 \leqslant \Delta = \frac{b}{a} \leqslant 1 \; , \tag{A-5}$$

$$
Q = \begin{vmatrix}
\frac{1}{3} + 2g_{11} & -4g_{13} & (\frac{1}{3} - g_{11})\Delta & -3g_{13}\Delta^3 \\
2g_{13} & -\frac{1}{7} - 4g_{33} & -g_{13}\Delta & (\frac{1}{7} - 3g_{33})\Delta^3 \\
(\frac{1}{3} - 2\ell_{11})\Delta^2 & 4\ell_{13}\Delta^4 & \frac{1}{3} + \ell_{11} & 3\ell_{13} \\
-2\ell_{13}\Delta^2 & (4\ell_{33} - \frac{1}{7})\Delta^4 & \ell_{13} & \frac{1}{7} + 3\ell_{33}
\end{vmatrix}
\tag{A-6}
$$

$$
g_{\mu\nu} = \sum_{m=1}^{\infty} \frac{4m+1}{2m} f_{2m}(\mu) f_{2m}(\nu) , \tag{A-7}
$$

$$
\ell_{\mu\nu} = \sum_{m=1}^{\infty} \frac{4m+1}{2m+1} f_{2m}(\mu) f_{2m}(\nu) , \qquad (\mu = 1, 3; \ \nu = 1, 3) , \tag{A-8}
$$

and the determinants Q_1^+, Q_2^+, Q_1 and Q_2 are obtained from Q by replacing respectively the first, second, third and fourth column with the column:

$$
\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \end{matrix} . \tag{A-9}
$$

In particular, when $\Delta = 0$ (see fig. 7) one has that

$$
\left(c_1^+\right)_{b=o} = \frac{5}{16} \zeta_2(kh)\, ka \, \frac{\frac{1}{7} + 4(g_{33} - g_{13})}{\left(\frac{1}{3} + 2g_{11}\right)\left(\frac{1}{7} + 4g_{33}\right) - 8g_{13}^2} \left\{ 1 + O\left[(ka)^2\right] \right\} , \tag{A-10}
$$

$$\left(c_2^+\right)_{b=o} = \frac{175}{16}\,\zeta_2(kh)(ka)^{-1}\,\frac{\frac{1}{3}+2(g_{11}-g_{13})}{\left(\frac{1}{3}+2g_{11}\right)\left(\frac{1}{7}+4g_{33}\right)-8g_{13}^2}\left\{1+0\left[(ka)^2\right]\right\}, \tag{A-11}$$

$$(d_1)_{b=o} = (d_2)_{b=o} = 0\,. \tag{A-12}$$

Substitution of relations (A-1) to (A-4) into formula (56-2) yields:

$$b_{2m} = \frac{5i(4m+1)}{8m(4m-1)!!}\,\zeta_2(kh)(ka)^{2m+3}\ \mathrm{x}$$

$$\mathrm{x}\left\{\frac{4}{5}\frac{2m+1}{(4m+1)!!}\frac{\zeta_{2m}(kh)}{\zeta_2(kh)}(ka)^{2m-2}+\frac{f_{2m}(1)}{6Q}\left(Q_1^+-Q_1\frac{\Delta}{2}\right)-\right. \tag{A-13}$$

$$\left.-\frac{f_{2m}(3)}{Q}\left(\frac{Q_2^+}{3}+Q_2\frac{\Delta^3}{4}\right)\right\}\left\{1+0\left[(ka)^2\right]\right\}\,.$$

where $(4m-1)!! = 1 \times 3 \times 5 \ldots \times (4m-1)$ is the semi-factorial of $(4m-1)$.

The Hertzian function U_4 in region IV is therefore given by (see formula (48)):

$$U_4 = U_0 - \frac{V}{24}M\,\frac{\zeta_2(kh)\,\zeta_2(kr)}{kh\,kr}\,(ka)^5\,(1+3\cos 2\theta)+0\left[(ka)^7\right], \tag{A-14}$$

where:

$$M = 1+\frac{25}{32Q}\left(\frac{1}{3}Q_2^+-\frac{1}{6}Q_1^++\frac{1}{12}Q_1\Delta+\frac{1}{4}Q_2\Delta^3\right), \tag{A-15}$$

and in particular:

$$(M)_{b=o} = \frac{168g_{11}g_{33}-54g_{13}^2-\frac{79}{16}g_{11}+\frac{175}{8}g_{13}+\frac{273}{16}g_{33}-\frac{233}{192}}{(1+6g_{11})(1+28g_{33})-54g_{13}^2}\,. \tag{A-16}$$

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FIGURE CAPTIONS

Fig. 1. Geometry for the scattering problem.

Fig. 2. Case $0 < h < b = a$.

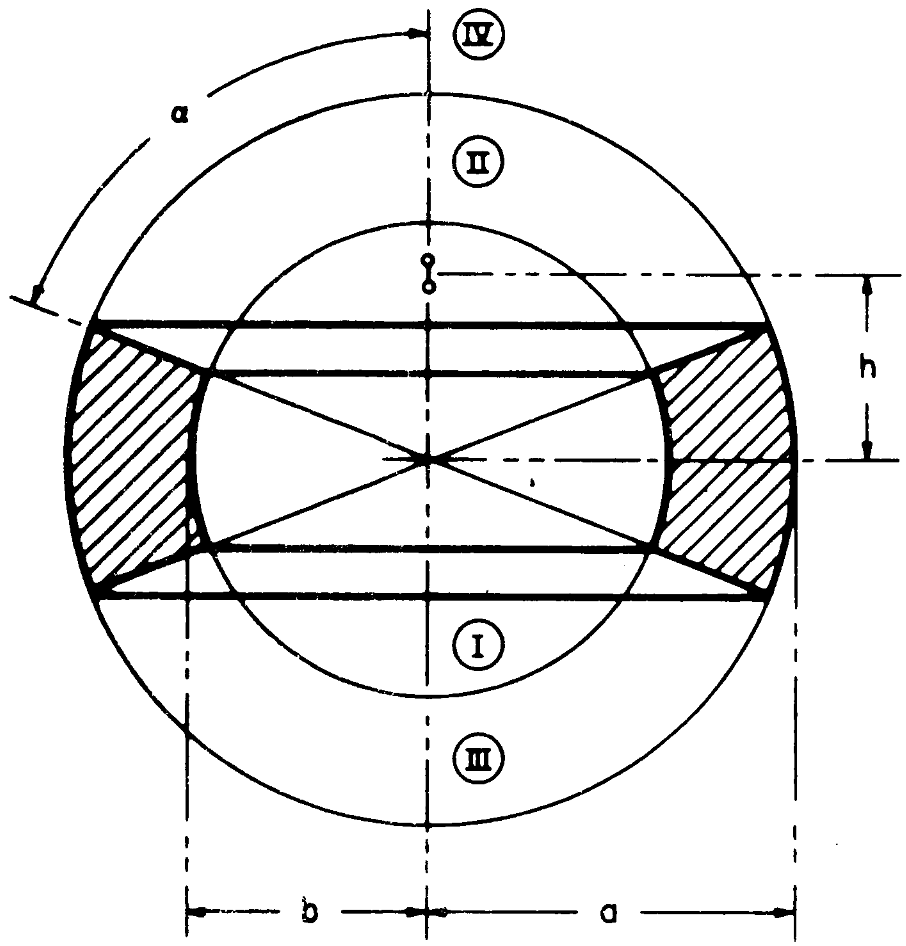
Fig. 3. Case $0 < h < b$, $\alpha = \pi/2$.

Fig. 4. Case $0 < h < b$, $a = \infty$.

Fig. 5. Case $0 < h < b$, $a = \infty$, $\alpha = \pi/2$.

Fig. 6. Case $h > a$, $b = 0$.

Fig. 7. Case $h > a$, $b = 0$, $\alpha = \pi/2$.



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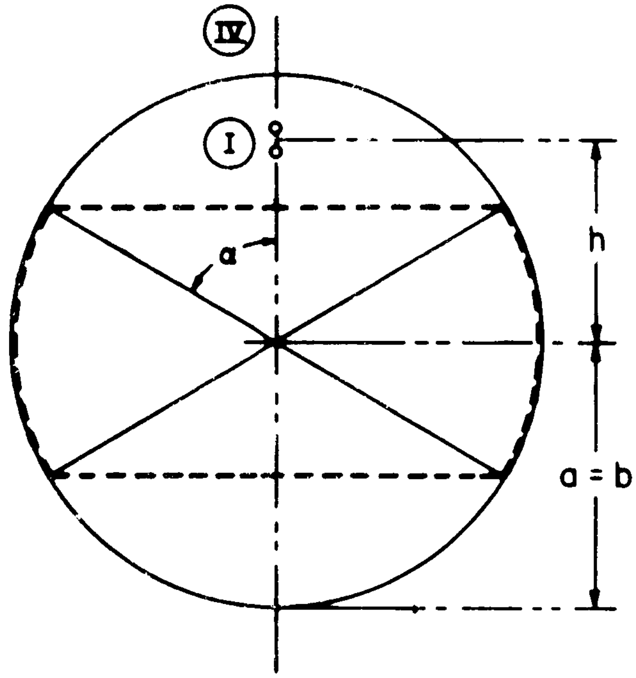
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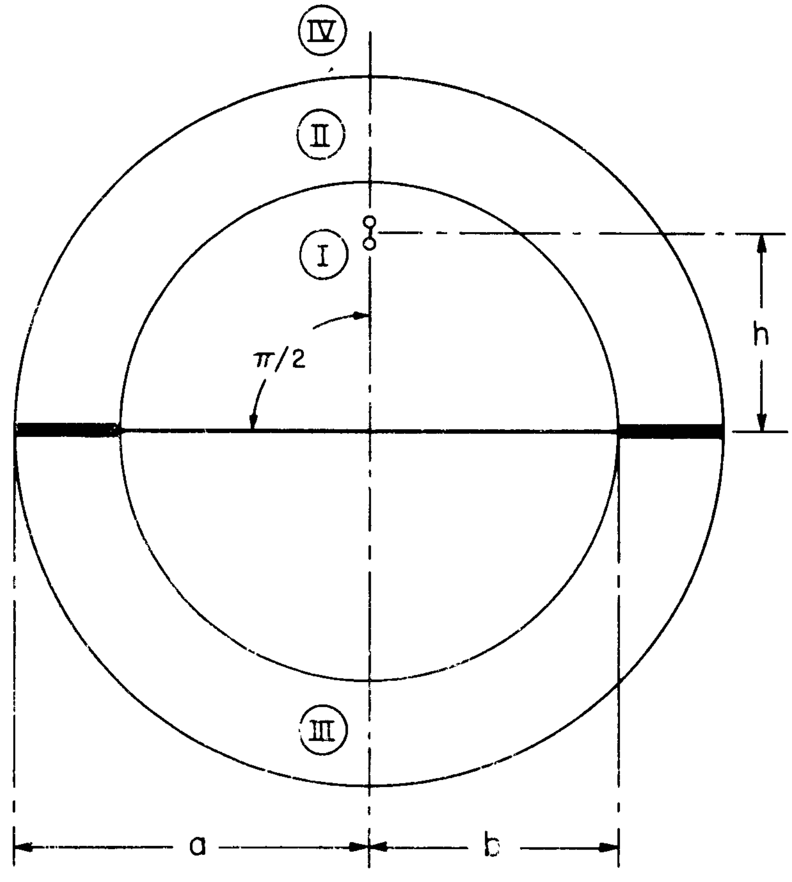
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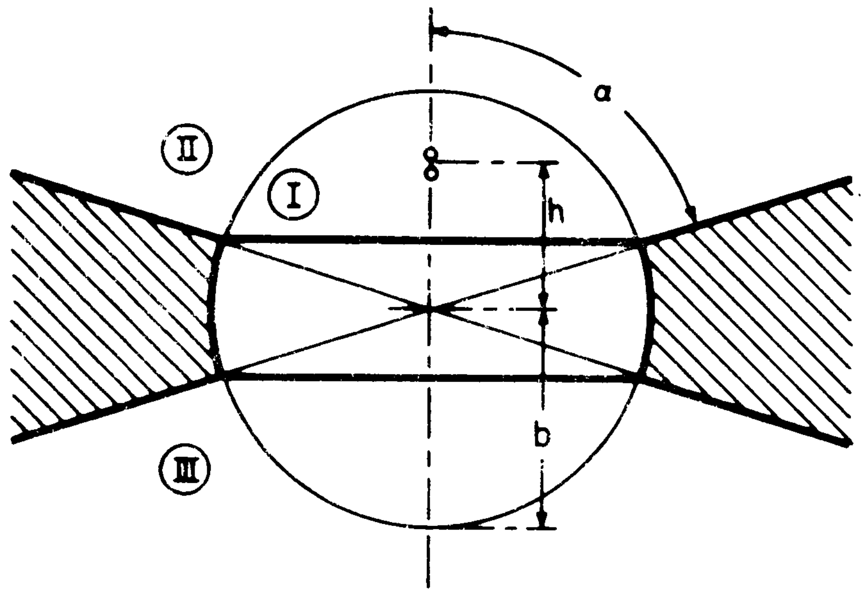

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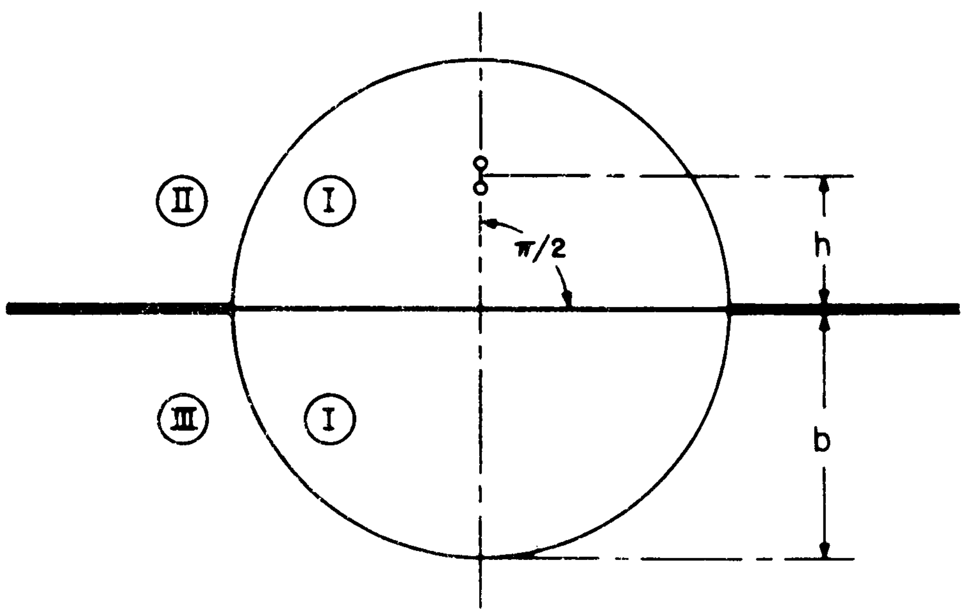

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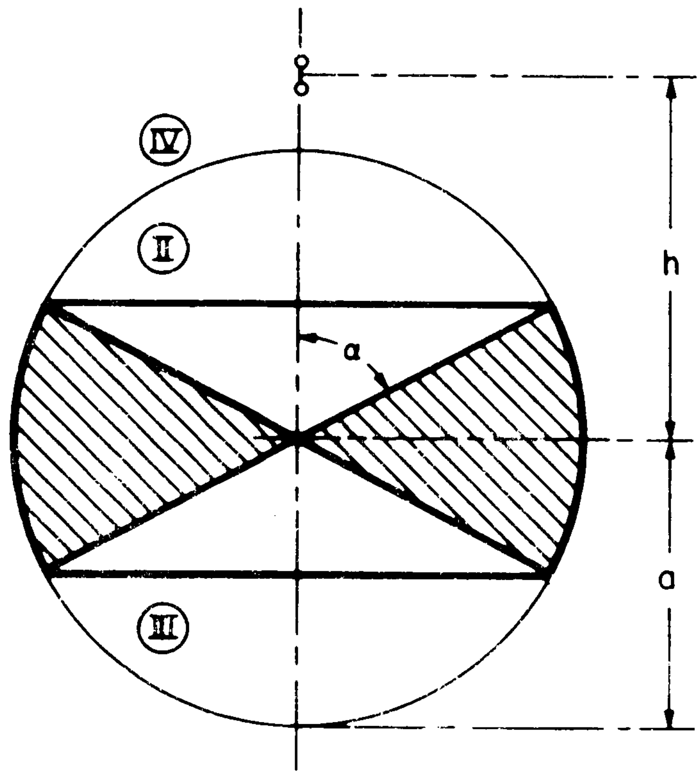
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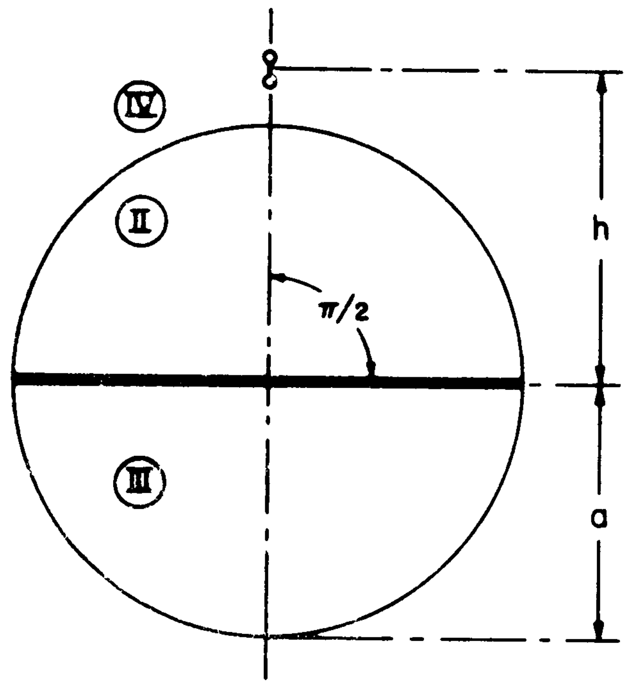
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